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New Results in the Theory of Differential Subordinations and Superordinations

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Abstract

In the theory of real-valued functions there are many important theorems that involve differential inequalities (i.e. inequalities containing the function and its derivatives), and the field of differential inequalities is a development of the last sixty years. For those which are interested in learning about differential inequalities, the recent texts by Lakshmikantham and Leela[1], Protter and Weinberger[2], and Walter[3], are highly recommended.

In the theory of complex-valued functions there are many differential implications in which the characterization of an analytic function is determined from a differential inequality. In this sense I mention only the well-known result of Noshiro[4]-Warschawski[5]-Wolff[6]: if f is analytic in the unit disk U, then

$$\mathrm{Re} f'(z) > 0,\ z \in \mathrm{U} \Rightarrow f \text{ is univalent in U.}$$

After this result involving first-order differential implications, these topics appeared only in two papers, that are in 1935 by G. Goluzin[7] and in 1947 by R. Robinson[8].

In the Geometric Function Theory (GFT) almost all of the geometric properties are described by differential inequalities, that are inequalities connecting the given analytic function and its derivatives. The complex differential inequalities deal with real-valued inequalities that involve the real or the imaginary part, or the module of a complex function and its derivatives (especially those of first and second order). The first article that deal with the *admissible functions method* appeared in the late of 1970's, and discuss about the second order inequalities in the complex plane (see[9]), while the second one appeared in 1981 (see[10]).

Many applications and the extensions of the theory of differential subordinations have been developed in the last period in different fields, like differential equations, partial differential equations, meromorphic functions theory, harmonic functions theory, integral operators theory, functions of several complex variables. As remarkably monographs in differential subordinations

[1]V. Lakshmikantham, S. Leela, *Differential and Integral Inequalities*, Academic Press, New York, 1969

[2]M. H. Protter, H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice Hall, Englewood Cliffs, New Jersey, 1967

[3]W. Walter, *Differential and Integral Inequalities*, Springer, New York, 1970

[4]K. Noshiro, *On the theory of schlicht functions*, J. Fac. Sci., Hokkaido Imp. Univ. Jap. (1), 2(1934-1935), 129-155

[5]S. E. Warschawski, *On the higher derivatives at the boundary in conformal mapping*, Trans. Amer. Math. Soc., 38(1935), 310-340

[6]J. Wolff, *L´intégrale d´une fonction holomorphe et à partie réelle positive dans un demi plan est univalente*, C. R. Acad. Sci. Paris, 198, 13(1934), 1209-1210

[7]G. M. Goluzin, *On the majorization principle in function theory*, Dokl. Akad. Nauk. SSSR, 42(1935), 647-650 (in Russian)

[8]R. M. Robinson, *Univalent majorants*, Trans. Amer. Math. Soc., 61(1947), 1-35

[9]S. S. Miller, P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., 65(1978), 298-305

[10]S. S. Miller, P. T. Mocanu, *Differential subordinations and univalent functions*, Michig. Math. J., 28(1981), 157-171

and superordination appeared recently I would like to mention the books [1], [2], and [3].

The present habilitation thesis is a survey about the most important contributions of the author in the last 13 years in the theory of *differential subordinations and superordinations*, and of its applications in the study of univalent and multivalent functions. Moreover, the applications in the theory of analytic integral and differential operators on analytic functions spaces have an important place in this thesis.

The first Chapter of the thesis, entitled *Sandwich-Type Theorems for Analytic Operators* deals to the beginning with classical results of those operators $\mathrm{I} : K \to H(\mathrm{U})$, $K \subset H(\mathrm{U})$, that satisfy

$$g(z) \prec f(z) \Rightarrow \mathrm{I}(g)(z) \prec \mathrm{I}(f)(z),$$

that are the *superordination-preserving* operators. I emphasize that almost all of the results included in this chapter are *sharp*, in the sense that under the given assumptions it cannot be improved.

The second paper that appeared in this field (but the first in chronological appearance) was published by the author, and this result combined by the related one about *subordination-preserving* operators gave a so called *sandwich-type* result for the integral operators of the form

$$A_{\beta,\gamma}(f)(z) = \left[\frac{\beta+\gamma}{z^{\gamma}} \int_0^z f^{\beta}(t) t^{\gamma-1} \,\mathrm{d}\, t\right]^{1/\beta}, \quad \beta, \gamma \in \mathbb{C}.$$

These studies have been continued by those related with *sandwich-type results for weighted integral operators*, and *superordinations-preserving general analytic operators*, that discussed this kind of behaviour for the operators having the form

$$\mathrm{I}_{h;\beta}[f](z) = \left[\beta \int_0^z f^{\beta}(t) h^{-1}(t) h'(t) \,\mathrm{d}\, t\right]^{1/\beta},$$

and

$$\varphi(p(z), zp'(z)) = \alpha(p(z)) + \beta(p(z))\, \gamma(zp'(z)),$$

respectively. I would like to emphasize that these results that appeared in not very high ranked journals have been cited more than 250 times till now.

Superodination-preserving and sandwich-type results are given in the next section for *generalized Briot-Bouquet differential operators*, in the following two cases, that are

$$\psi(p(z), zp'(z)) = \alpha(p(z)) + \beta(p(z))\gamma(zp'(z))$$

or

$$\psi(p(z), zp'(z)) = \alpha(zp'(z)) + \beta(zp'(z))\gamma(p(z)).$$

The Briot-Bouquet differential subordination plays a very important role in GFT, especially in the univalent function theory, since the analytic solution of the Briot-Bouquet differential equation is closely connected with the Bernardi-Libera-Livingston integral operators. At its turn, these last one type of operators have been extensively studied by many authors and have a crucial role in our field of interest.

Very recently, we obtained sandwich-type results for a wide class of convex integral operators of the form

$$A^{\phi,\varphi}_{\alpha,\beta,\gamma}[f](z) = \left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)} \int_0^z f^{\alpha}(t)\varphi(t) t^{\delta-1} \,\mathrm{d}\, t\right]^{1/\beta}.$$

The most important remark related with this last one, is the fact that the useful tool of *Loewner chains* used in many proof was completed to the fact that these chains need to represent a normal family. Otherwise, the proofs are not complete, and a such of counterexample I showed at many conferences. Starting with this last mentioned result (paper), all the further proofs for similar results need to follow the steps presented in the proof of the main result. Taking into the account the importance of this result (mainly, of the proof), the entire proof contained in my paper is included in the thesis (see Section 1.5).

In the same way, we improved this last result by a better one, in a joint paper with N. E. Cho and H. R. Srivastava, that is presented in the next section.

In a paper that appeared this year, sandwich-results for integral operators of the form

$$\mathrm{I}^n_{h;\beta,\alpha_i}\left[f_i\right](z) = \left[\frac{\sum_{i=1}^n \alpha_i}{z^{\sum_{i=1}^n \alpha_i - \beta}} \int_0^z \left(\prod_{i=1}^n f_i^{\alpha_i}(t)\right) h^{-1}(t)h'(t)\, dt\right]^{\frac{1}{\beta}}$$

are given. The results have been difficultly obtained, since the analyticity of the operators doesn't follow straightly from the well-known *Integral Existence Theorem* of S. S. Miller and P. T. Mocanu. Moreover, the 'weights' that appeared in this case are different than those of the related previous cases.

This chapter ends with sandwich-type results for generalized Srivastava-Attiya integral operators, connected with the Hurwitz-Lerch Zeta functions. I would like to underline the slight, relatively weak assumptions of the main results, for which these operators preserve the subordinations and the superordinations.

The second Chapter, entitled *Applications to the Study of Analytic Functions*, contains many recent applications of the differential subordination (and/or superordination) theory to the study of subclasses of univalent and multivalent functions.

Thus, the first section shows us inclusion properties, and sandwich-type results for subclasses of p-valent functions defined by the new Liu-Owa linear operator, i.e.

$$Q^{\alpha}_{\beta,p}f(z) = \left(\begin{array}{c} p+\alpha+\beta-1 \\ p+\beta-1 \end{array}\right) \frac{\alpha}{z^{\beta}} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\,\mathrm{d}\,t,\ \text{for } \alpha>0,\ \beta>-1,$$
$$Q^{0}_{\beta,p}f(z) = f(z),\ \text{for } \alpha=0,\ \beta>-1.$$

The next section contains the study of some classes meromorphic functions defined by a multiplier transformation of the form

$$\mathrm{I}^m_p(n;\lambda,l)f(z) = (\Phi^{p,m}_{n;\lambda,l} * f)(z), \quad \text{where} \quad \Phi^{p,m}_{n;\lambda,l}(z) = z^{-p} + \sum_{k=n}^{\infty}\left[\frac{\lambda(k+p)+l}{l}\right]^m z^k.$$

We obtained containment relations between the subclasses of these function, we discussed radius problems, and we determined the behaviour of the images of these functions by the integral operator

$$F_{p,c}(f)(z) = \frac{c}{z^{c+p}} \int\limits_0^z t^{c+p-1} f(t)\,\mathrm{d}\,t,\ (c>0).$$

In another recent article, we studied subordination results for a subclass of meromorphic functions defined by the convolution operator

$$D^m_{\lambda,p}f(z) = D^1_{\lambda,p}\left(D^{m-1}_{\lambda,p}f(z)\right) = z^p + \sum_{k=n}^{\infty}\left(\frac{p+\lambda k}{p}\right)^m a_{k+p}z^{k+p},\ m\in\mathbb{N}.$$

Inclusion properties, radius problems, and the images of these functions by the integral operator

$$F_{\delta,p}(f)(z) = \frac{\delta+p}{z^{\delta}} \int_0^z t^{\delta-1} f(t)\,\mathrm{d}\,t,\ z\in\mathrm{U} \quad (\delta>-p)$$

are studied. In both of these last two sections, many interesting applications for special cases are given.

The study of subclasses of multivalent functions involving the generalized Jung-Shams operator, i.e.

$$\mathrm{I}_p^{\alpha} f(z) = \frac{(p+1)^{\alpha}}{z\Gamma(\alpha)} \int\limits_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)\, dt, \quad \text{for} \quad \alpha > 0,$$
$$\mathrm{I}_p^{0} f(z) = f(z), \quad \text{for} \quad \alpha = 0,$$

is given in the following section. We obtained several properties of the convolution product of the functions belonging to these classes, more exactly information about the image of these products by the I_p^{α} operator. The special cases are very interesting, some lower bounds of the functionals have been obtained using the well-known lemma of Wilken and Feng, and the results were not trivial, even very hard to obtain (as the reviewers mentioned).

The next section deals with a subclasses of multivalent functions involving the extended fractional differintegral operator

$$\Omega^{(\lambda,p)} f(z) = z^p + \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1)\,\Gamma(p+1-\lambda)}{\Gamma(p+1)\,\Gamma(k+p+1-\lambda)}\, a_{k+p}\, z^{k+p}$$
$$= z^p\ {}_2F_1(1, p+1, p+1-\lambda; z) * f(z),\ z \in \mathrm{U}.$$

We discussed inclusion relations, radius problems, and the properties of these functions images by the differintegral operator

$$\mathrm{F}_{\mu,p}(f)(z) = \frac{\mu+p}{z^{\mu}} \int_0^z t^{\mu-1} f(t)\,\mathrm{d}\,t = \left(z^p + \sum_{k=1}^{\infty} \frac{\mu+p}{\mu+p+k} z^{p+k}\right) * f(z)$$
$$= z^p\ {}_2F_1(1, \mu+p, \mu+p+1; z) * f(z),\ z \in \mathrm{U}.$$

Also, we gave many special cases for our main results.

In the last section of this chapter, starting from a result related to the Briot-Bouquet differential subordination, we obtained very recently two theorems involving a class of general operators that satisfy a given differential condition. These results generalize many of those previously obtained by many authors, and correct some other ones.

Like to the beginning of the description of the first Chapter, I underline that almost all of the results included in this chapter are *sharp*, in the sense that under our given assumptions these implications cannot be improved.

The third Chapter, entitled *New Results in the Geometric Theory of Analytical Functions*, contains some of my results connected with the classic theory of univalent functions.

Thus, in the first section we gave simple sufficient condition for starlikeness of some order, that generalize or are connected with some previous results of Li and Owa and Lewandowski, Miller and Zlotkiewicz, respectively. Moreover, using a result of M. Robertson we extended a few well-known results of Owa, Obradović and Lee. We remark that all these results are very simple to be used in some given calculatory problems, and the assumptions could be very easily checked.

In the second section, using the method of differential subordinations we investigate inclusion relationships among two subclasses of p-valent functions defined by means of Liu–Owa operator $Q^{\alpha}_{\beta,p}$ that was previously mentioned. The properties of the images of these two type of functions by a special integral operator is studied, together with radius problems, while all the bound obtained and all the dominant are the best possible; the results improve those obtained by Liu.

Next, using the method of differential subordinations, certain inclusion relations between the class of α-convex type functions and a class of Bazilević type functions are obtained. We studied the connections between two subclasses of analytic functions: the first one generalize the alpha-convex functions, while the second one represents a subclass of Bazilević type functions. We gave many special cases that could be very easily be used, since the assumptions are surprisingly simple to be checked.

The following section contains interesting results about subclasses of spirallike multivalent functions. Thus, we studied convolution properties of spirallike starlike and convex functions, and some special cases of the main results are also pointed out. Further, we also obtained inclusion and convolution properties for some new subclasses on p–valent functions defined by using the Dziok–Srivastava operator. Remark that these type of results we gave here, that deal with the fact that some power series (i.e. analytic functions) has no zeros in the unit disk for some values of the parameters, was neglected till now, and no too many related studies are known.

In the last section, that is one of my newest result, we obtain generalized sufficient conditions for the starlikeness and convexity of some normalized analytic functions, that results extend those recently obtained in the work of Uyanik and Owa, moreover it correct their result. Since the proof of the previously mentioned paper seems to be logically unclear, we tried to improve and extend some simple conditions for starlikeness and convexity for the functions with missing coefficients. By using some simple integral inequalities, we tried to generalize some previous results that seems to be not correct (mainly, to correct it), and we gave some interesting special cases of our results. The methods used here are frequently used in complex analysis and operator theory on spaces of analytic functions. Among others, for example, in obtaining estimations for point evaluation operators on spaces of analytic functions of one or several variables containing derivatives in the definition of the spaces (see, for example, some papers of S. Stević). Since the proof seems to be a special one, I included it in the thesis.

The fourth Chapter has the title *Miscellaneous Contributions in the Theory of Univalent Function*. The first section represents an attempt to extend some results about *averaging integral operators* for the operators

$$\mathrm{I}_{h;\beta,\gamma}[f](z)=\left[\frac{\gamma}{h^{\gamma}(z)}\int_0^z f^{\beta}(t)h^{\gamma-1}(t)h'(t)\mathrm{d}t\right]^{1/\beta}.$$

The novelty consists also in the proof of the main result, where one of the contact boundary point was considered as fixed, and the Loewner chain was defined with the aid of this point.

The second result is connected with the notion of *quasi-subordination*, and/or *majorization*. This concept is weaker than those of subordination, but in some cases we could still find good results in this area. Thus, for a class of functions defined by the multiplier transformation

$$\mathcal{I}_p(n,\lambda)f(z)=z^p+\sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k,\quad (\lambda\in\mathbb{C}\setminus\{-p\},\ n\in\mathbb{Z}),$$

we gave a majorization results between two operators belonging to the above classes. The results can be applied very easily for many special cases, and in particular, we get some earlier well-known results.

The chapter and the thesis ends with my last result that represents the study of many properties of *bi-univalent functions*. These functions have been introduced in a paper appeared in Studia Universitatis Babeş-Bolyai – Mathematica in 1986, and many articles related to these classes appeared recently. We showed that this class is preserved under the main transformations, we found the covering, the distortion and the rotation theorems. Moreover, we gave a growth theorem and a combined growth and distortion theorem. We conclude that, in the case that Brannan and Clunie's conjecture could be proved affirmatively, our theorems will have a better estimates. Till now, I didn't know that these kind of investigation has be done for these classes.

Reference monographs

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker, New York, 1999

[2] P. T. Mocanu, T. Bulboacă, Gr. Şt. Sălăgean, *Geometric Function Theory of Univalent Functions* (in Roum.), House of Science Book Publ., Cluj-Napoca, 1999, 410+vii pag.

- Second edition (revised and completed), House of Science Book Publ., Cluj-Napoca, 2006, 460+ix pag.

[3] T. Bulboacă, *Differential Subordinations and Superordinations. New Results*, House of Scientific Boook Publ., Cluj-Napoca, 2005, 297+vi pag., ISBN 973-686-777-3.

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